Time-optimal control in $SU(2)$ and $SO(3)$.

Yuly Billig[1]

Abstract. Spatial orientation of a solid body is parametrized by the group SO_3 of rotations of a sphere. If we are given two types of control, that is we are able to rotate the body around two given axes, then we can achieve an arbitrary orientation of this body. In this paper we discuss how to perform this operation in the *optimal* way, minimizing the sum of the angles of rotation. We hope that our methods and results can be applied to improve control of the Kepler spacecraft using two reaction wheels.

0. Introduction.

The present paper is a shortened version of [1]. Paper [1] was written with a view towards applications to quantum computing and quantum control, and is devoted to optimal control in the group SU_2. However the group SU_2 is closely related to the group SO_3 of rotations of the sphere. More precisely, SU_2 is a double cover of SO_3, i.e. there exists a 2:1 group homomorphism

$$SU_2 \to SO_3.$$

As a consequence, the results of [1] are also applicable to the group of rotations.

It is well-known that rotation matrices can be written as exponentials of skew - symmetric matrices, for example the matrix of rotation in angle t around Z-axis can be written in the following way:

$$\begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp\left(t \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right).$$

Suppose we are given two linearly independent skew-symmetric matrices X and Y. Their exponentials $\exp(tX)$ and $\exp(tY)$ are rotations around two axes specified by X and Y. Suppose that these two kinds of rotations are the two available controls. In this case any spatial rotation $g \in SO_3$ can be decomposed in a product

$$g = \exp(t_1 C_1) \times \ldots \times \exp(t_n C_n), \tag{0.1}$$

where $C_i \in S = \{X, Y\}$, $t_i \in \mathbb{R}$.

The problem of finding explicit factorizations of type (0.1) goes back to Euler [2]. The exponentials $\exp(tC_i)$ corresponding to

$$C_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

[1] School of Mathematics and Statistics, Carleton University, 1125 Colonel By Drive, Ottawa, K1S 5B6, Canada. E-mail: billig@math.carleton.ca

2010 Mathematics Subject Classification: 81P68, 22E70, 57R27.

are the rotations in angle t around X- and Y-axis respectively. Euler proved that every $g \in SO_3$ can be factored as $g = \exp(t_1 C_1)\exp(t_2 C_2)\exp(t_3 C_1)$ for some $t_1, t_2, t_3 \in [0, 2\pi)$. The parameters t_1, t_2, t_3 are called Euler angles.

Given a rotation $g \in SO_3$ and two controls X, Y, we pose the following:

Optimal control problem: Among all decompositions (0.1) find the one that minimizes the sum of the angles of rotation: $|t_1| + \ldots + |t_n|$.

Below we present the results of [1], where this problem was solved for a closely related group SU_2 and with an additional assumption that all $t_i > 0$ (the paper [1] is tailored towards control of quantum systems, where t_i represent time and thus must be positive).

The methods of [1] can be easily modified for the group SO_3 and the assumption on positivity of t_i may be dropped, with only a minor modification of the results.

As we shall see in this paper, this optimization problem, as posed, might not have a solution – there may be a sequence of factorizations of type (0.1) with the number of factors n going to infinity, while the total time going to infinum. For this reason we slightly modify the above optimization problem and ask for the infinum of total times for all factorizations of a given $g \in SU_N$

In this paper we solve this problem for the group SU_2 with the set of controls S consisting of two elements, $S = \{X, Y\}$.

We show that the infinum time does not change if we replace $S = \{X, Y\}$ with its convex closure $\overline{S} = \{\tau X + (1-\tau)Y \,|\, 0 \le \tau \le 1\}$. In the latter case there will be in fact an optimal decomposition (0.1) with a finite number of factors. It turns out that to get an optimal decomposition it is sufficient to add to S at most one element $W \in \overline{S}$, the one that is orthogonal to $X - Y$.

The new control $\tau X + (1-\tau)Y$ has the meaning of turning on both controls X and Y simultaneously with relative intensities (speeds) given by the ratio of τ and $1 - \tau$.

In this paper we give explicit descriptions of the optimal decompositions in SU_2. Since there is a surjective homomorphism $SU_2 \to SO_3$, our results are also applicable to the group SO_3.

Our solution is based on the method of Lagrange multipliers adopted for the set-up of Lie groups. Alternatively one could use the geometric theory developed in [3], which is based on the Pontryagin maximum principle. The general methods, however, give only necessary conditions for optimality, which in practice are not sufficient. To get the desired results we supplement these general methods with some explicit calculations in SU_2 which allow us to get stronger and more explicit optimality conditions.

For the sake of brevity, we omit the proofs and refer to [1], where all the proofs are given.

Acknowledgements: I am thankful to Velimir Jurdjevic and Alexander Weekes for stimulating discussions. This work is supported with a grant from the Natural Sciences and Engineering Research Council of Canada.

1. Time-optimal decompositions in Lie groups.

Let G be a compact connected real Lie group, and let $\mathfrak{g}$ be its Lie algebra. An element $X \in \mathfrak{g}$ defines a 1-parametric subgroup $\{\exp(tX) | t \in \mathbb{R}\}$. In this paper we study an optimal control problem on G, describing optimal decompositions of an arbitrary given element $g \in G$ into a product of exponentials $\exp(tX)$ with X belonging to a fixed set S of controls, $S \subset \mathfrak{g}$ and positive times t.

The following well-known criterion describes when the group G is controllable by a set S:

Theorem 1.1. ([3], Theorem 6.1) A real connected Lie group G is generated by its subgroups $\{\exp(tX)\}$, $X \in S$, if and only if S generates the Lie algebra $\mathfrak{g}$.

This theorem is proved using topological methods, and does not provide an effective way of finding such decompositions. In this context it is natural to pose the problem of describing decompositions that are time-optimal:

Problem. For a given $g \in G$ determine

$$\inf \left\{ t_1 + \ldots + t_n \,\middle|\, g = \exp(t_1 C_1) \cdot \ldots \cdot \exp(t_n C_n),\ t_i \geq 0, C_i \in S \right\}. \tag{1.1}$$

A compact Lie group G is isomorphic to a Lie subgroup in a general linear group ([4], Corollary 4.22), so we assume that $G \subset GL_d(\mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. The Lie algebra $\mathfrak{g}$ is then a real subalgebra in the matrix Lie algebra $M_d(\mathbb{F})$. We fix an $\mathbb{R}$-bilinear positive-definite G-biinvariant scalar product $\langle \cdot, \cdot \rangle$ on $M_d(\mathbb{F})$, which induces a norm on $M_d(\mathbb{F})$ with the property

$$|AB| \leq |A| \cdot |B| \quad \text{for all} \quad A, B \in M_d(\mathbb{F}).$$

Theorem 1.2. Optimization problem (1.1) in SU_2 with $S = \{X, Y\}$ is equivalent to the problem with the set of controls $\overline{S} = \{\tau X + (1 - \tau) Y \,|\, \tau \in [0, 1]\}$.

Since a uniform rescaling of the controls $X' = cX$, $Y' = cY$, gives an equivalent optimization problem (with optimal time rescaled by a factor c^{-1}), we may assume without loss of generality that $|X| = 1$ and $|Y| \geq |X|$. Let $\kappa = 1/|Y| \leq 1$. We may pass to the normalized set of controls $\{X, Y/|Y|\}$ by replacing the total time $t_1 + \ldots + t_n$ for the decomposition $\exp(t_1 C_1) \cdot \ldots \cdot \exp(t_n C_n)$ with the cost function

$$\sum_{i=1}^{n} \kappa_i t_i, \quad \text{where } \kappa_i = \begin{cases} 1, & \text{if } C_i = X, \\ \kappa, & \text{if } C_i = Y/|Y|. \end{cases} \tag{1.2}$$

From now on we assume that $S = \{X, Y\}$ with $|X| = |Y| = 1$ and consider the optimization problem with cost function (1.2) where the *cost factor* $\kappa \leq 1$. This will allow us to consider the limiting case $\kappa = 0$, when there is no cost associated with control Y.

Let us introduce some terminology.

An *admissible word of length* n is an expression $\exp(t_1 C_1) \cdot \ldots \cdot \exp(t_n C_n)$ with $n \geq 0$, $t_i \geq 0$ and $C_i \in S$. A word of zero length is the identity element of G.

Every admissible word can be written in a *reduced form*, where $t_i > 0$ and $C_i \neq C_{i+1}$ for all i.

A decomposition of $g \in G$ as an admissible word of length n is called *n-optimal* if it has the minimum cost among all admissible words of length n that are equal to g.

A decomposition of $g \in G$ as an admissible word of length n is called *optimal* if it has the minimum cost among all admissible words of arbitrary lengths that are equal to g.

For a given $g \in G$, an optimal decomposition may not exist since there might be a sequence of decompositions of g of increasing lengths and with cost going to infinum. On the other hand, n-optimal decompositions exist, as we show in the following Lemma.

Lemma 1.3. Let G be a connected Lie group, and suppose that the set of controls S is finite and satisfies the following condition: every generator $C \in S$ with a zero cost factor $\kappa = 0$ has a periodic exponential, i.e., $\exp(TC) = 1$ for some $T > 0$. If $g \in G$ has a decomposition as an admissible word of length n then it has an n-optimal decomposition.

The following Lemma is obvious:

Lemma 1.4. (a) If a word of length n is optimal then it is n-optimal.

(b) If the word

$$\exp(t_1 C_1) \cdot \ldots \cdot \exp(t_n C_n) \tag{1.3}$$

is optimal (resp. n-optimal), then its subword

$$\exp(t_p C_p) \cdot \ldots \cdot \exp(t_k C_k)$$

with $1 \leq p \leq k \leq n$ is also optimal (resp. $k - p + 1$-optimal).

(c) If the word (1.3) is optimal then

$$\exp(s_1 C_1) \exp(t_2 C_2) \cdot \ldots \cdot \exp(t_{n-1} C_{n-1}) \exp(s_n C_n)$$

with $0 \leq s_1 \leq t_1$, $0 \leq s_n \leq t_n$, is also optimal.

The last claim of the lemma suggests that for the optimal words there are stronger constraints on time parameters t_i with $2 \leq i \leq n-1$. We will call these the *middle time parameters*.

2. Optimal words in SU_2.

For the rest of the paper we will focus on the case $G = SU_2$. We find it convenient to use the realization of SU_2 as the unit sphere in the quaternion algebra $\mathbb{H}$:

$$SU_2 = \left\{ a1 + bi + cj + dk \,|\, a^2 + b^2 + c^2 + d^2 = 1 \right\}.$$

The Lie algebra su_2 in this realization is the tangent space at 1 and has basis $\{i, j, k\}$. This basis is orthonormal relative to the invariant bilinear form. The norm in $\mathbb{H}$ satisfies $|xy| = |x| \cdot |y|$ and is SU_2-bi-invariant.

The isomorphism with the standard matrix construction of su_2 is given by Pauli matrices:

$$i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The group SU_2 acts on its Lie algebra by the conjugation automorphisms. Since $-I$ acts trivially, this action factors through $SO_3 \cong SU_2 / \{\pm I\}$, and is given by the natural action of SO_3 on $\mathbb{R}^3$.

The action of SO_3 on the unit sphere is transitive, so without loss of generality we may assume that $X = i$, while $Y = i\cos\alpha + j\sin\alpha$, where α is the angle between the vectors X and Y, $0 < \alpha < \pi$. This identification will allow us to carry out certain calculations in an explicit form. If C is an element of su_2 of norm 1 then $\exp(tC) = \cos(t) + C\sin(t)$.

Since $\exp(\pi X) = \exp(\pi Y) = -1$ is a central element, we see that an n-optimal word satisfies the following

π-Condition: at most one time parameter may be greater or equal to π; without loss of generality we may assume that this parameter corresponds to the generator Y, as it has a lower cost, and is not a middle time parameter.

We begin by describing 4-optimal words.

Proposition 2.1. (a) Let

$$g = \exp(t_1X)\exp(t_2Y)\exp(t_3X)\exp(t_4Y) \tag{2.1}$$

be a 4-optimal word with $t_1, t_2, t_3 < \pi$. Then either

$$\frac{\tan(t_2)}{\tan(t_3)} = \frac{\kappa - \cos(\alpha)}{1 - \kappa\cos(\alpha)} \tag{2.2}$$

or (2.1) is not reduced.

(b) The same condition holds for a 4-optimal word

$$\exp(t_4Y)\exp(t_3X)\exp(t_2Y)\exp(t_1X).$$

Remark 2.2. The denominator of $\frac{\kappa-\cos(\alpha)}{1-\kappa\cos(\alpha)}$ is non-zero since $0 \leq \kappa \leq 1$ and $|\cos(\alpha)| < 1$. If the numerator of this fraction vanishes, the condition (2.2) should be replaced with

$$(1 - \kappa\cos(\alpha))\sin(t_2)\cos(t_3) = (\kappa - \cos(\alpha))\cos(t_2)\sin(t_3). \tag{2.3}$$

Using Proposition 2.1 and Lemma 1.4 we get a description of n-optimal words:

Corollary 2.3. Suppose $\kappa \neq \cos(\alpha)$. Let $\exp(t_1C_1)\exp(t_2C_2)\cdot\ldots\cdot\exp(t_nC_n)$ be a reduced n-optimal word with $n \geq 4$. Then

$$t_p = t_{p+2} \text{ for all } \ 2 \leq p \leq n-3. \tag{2.4}$$

Since in an n-optimal word all middle parameters corresponding to the same control are equal, we will denote by t_x (resp. t_y) the middle time parameters corresponding to X (resp. Y).

Corollary 2.4. Under the assumptions of the previous corollary,

$$\frac{\tan(t_y)}{\tan(t_x)} = \frac{\kappa - \cos(\alpha)}{1 - \kappa\cos(\alpha)}. \tag{2.5}$$

We see from Corollaries 2.3 and 2.4 that reduced n-optimal words are described with at most three independent time parameters for all n. Since the group SU_2 is three-dimensional, we conclude that for each n there exists only a finite number of n-optimal words representing a given $g \in SU_2$ (for the case $\kappa = \cos(\alpha)$ see Theorem 2.12 below).

Next we shall investigate optimality of words of length 3.

Proposition 2.5. Let $\cos(t) > 0$ and let $\epsilon > 0$ be a small parameter. Then

$$(i) \quad \exp(\epsilon X)\exp(tY)\exp(\epsilon X) = \exp(\tau Y)\exp(\mu X)\exp(\tau Y) \tag{2.6}$$

and

$$(ii) \quad \exp(\epsilon Y)\exp(tX)\exp(\epsilon Y) = \exp(\tau X)\exp(\mu Y)\exp(\tau X), \tag{2.7}$$

where

$$\tau = t/2 + \epsilon\cos(\alpha)(1 - \cos(t)) + o(\epsilon) \tag{2.8}$$

and

$$\mu = 2\epsilon\cos(t) + o(\epsilon). \tag{2.9}$$

Corollary 2.6. Let $0 < t < \frac{\pi}{2}$ and let $\epsilon > 0$ be a small parameter. Then
(i) The word $\exp(\epsilon X)\exp(tY)\exp(\epsilon X)$ is not 3-optimal.
(ii) If $\kappa > \cos(\alpha)$ then the word $\exp(\epsilon Y)\exp(tX)\exp(\epsilon Y)$ is not 3-optimal.

Corollary 2.7. Suppose $\kappa > \cos(\alpha)$. A reduced optimal word $\exp(t_1 C_1)\ldots\exp(t_n C_n)$ of length $n > 1$ satisfies the *strong π-condition*, which is the π-condition with an additional restriction $t_1, t_n \leq \pi$.

Now we can describe possible optimal decompositions in SU_2. We will consider several cases.

Theorem 2.8. Suppose $\cos(\alpha) < \kappa \leq 1$. Then the infinum of the cost of admissible decompositions of a given element of SU_2 is attained either

(a) on a reduced n-optimal word

$$\exp(t_1 C_1)\exp(t_2 C_2)\cdot\ldots\cdot\exp(t_n C_n)$$

satisfying the strong π-condition, (2.4) with $\frac{\pi}{2} \leq t_x < \pi$, $\frac{\pi}{2} \leq t_y < \pi$, and when $n \geq 4$ the condition (2.5).
or

(b) on a word

$$\exp(t_1 C_1)\exp(t_2 W)\exp(t_3 C_3),$$

where $C_1, C_3 \in \{X, Y\}$, $t_1, t_2, t_3 \geq 0$ and

$$W = (1 - \kappa\cos(\alpha))X + (\kappa - \cos(\alpha))Y$$

with the cost of $\exp(t_2 W)$ equal to $(\kappa^2 - 2\kappa\cos(\alpha) + 1)t_2$.

Remark 2.9. The vector W is orthogonal to the line passing through X and Y/κ.

Remark 2.10. Time-optimal decompositions that appear in Theorem 2.8 involve at most three independent time parameters. Since the group SU_2 is 3-dimensional, there will be a finite number of such decompositions of each length. Moreover, since middle times in the decomposition (a) above are at least $\frac{\pi}{2}$, any given decomposition of g gives a bound on the length of an optimal decomposition of type (a).

Theorem 2.11. Suppose $0 < \kappa < \cos(\alpha)$. Then the infinum of the cost of admissible decompositions of a given element of SU_2 is attained on a reduced n-optimal word

$$\exp(t_1 C_1) \exp(t_2 C_2) \cdot \ldots \cdot \exp(t_n C_n)$$

satisfying the π-condition, (2.4) with $\frac{\pi}{2} \leq t_y < \pi$, and when $n \geq 4$ the conditions $0 < t_x \leq \frac{\pi}{2}$ and (2.5).

Theorem 2.12. Suppose $\kappa = \cos(\alpha) > 0$. Then the infinum of the cost of admissible decompositions of a given element of SU_2 is attained on a reduced n-optimal word

$$\exp(t_1 C_1) \exp(t_2 C_2) \cdot \ldots \cdot \exp(t_n C_n)$$

satisfying the π-condition, (2.4) with $\frac{\pi}{2} \leq t_y < \pi$, and when $n \geq 4$ the condition $t_x = \frac{\pi}{2}$.

Finally let us consider the case when the cost associated with the generator Y is zero. In this case we have the freedom of replacing the generator Y with $-Y$ since $\exp(-tY) = \exp((2\pi - t)Y)$. Thus without loss of generality we may assume that $\cos(\alpha) \leq 0$. The case when $\cos(\alpha) < 0$ is then covered by Theorem 2.8.

Theorem 2.13. Let $\cos(\alpha) = 0$ and $\kappa = 0$. For any $g \in SU_2$ the infinum cost is attained on a word of length at most 3.

In conclusion, let us consider a particular case.

Example 2.14. Let $X = i$, $Y = j$ and $\kappa = 1$. Then an optimal decomposition of any $g \in SU_2$ is given by words of the following types:

$$(a) \exp(t_1 C_1), \text{ where } 0 \leq t_1 < 2\pi;$$

$$(b) \quad \exp(t_1 C_1) \exp(t_2 C_2), \text{ where } 0 < t_1, t_2 \leq \pi;$$

$$(c) \quad \exp(t_1 C_1) \exp(t_2 C_2) \exp(t_3 C_1), \text{ where } t_2 \geq \frac{\pi}{2}, \ t_1, t_2, t_3 \leq \pi;$$

$$(d) \quad \exp(t_1 C_1) \exp(t_2 C_2) \exp(t_2 C_1) \exp(t_3 C_2), \text{ where } t_2 \geq \frac{\pi}{2}, \ t_1, t_2, t_3 \leq \pi;$$

or

$$(e) \quad \exp(t_1 C_1) \exp\left(t_2 \frac{i + j}{2}\right) \exp(t_3 C_3),$$

where $C_i \in \{X, Y\}$, $C_1 \neq C_2$, with the infinum time $\sum_k t_k$. Here $\exp\left(t\frac{i+j}{2}\right)$ may be viewed as

$$\exp\left(t\frac{i+j}{2}\right) = \lim_{N\to\infty}\left[\exp\left(\frac{ti}{2N}\right)\exp\left(\frac{tj}{2N}\right)\right]^N.$$

References:

[1] Y. Billig, Time-optimal decompositions in SU(2), Quantum Information Processing **12** (2013), 955-971.
[2] L. Euler, Formulae generales pro translatione quacunque corporum rigidorum, Novi Commentarii Academiae Scientiarum Petropolitanae **20** (1776), 189-207.
[3] V. Jurdjevic, Geometric control theory, Cambridge studies in advanced mathematics **51**, Cambridge University Press, New York, 1997.
[4] A. W. Knapp, Lie groups beyond an introduction, Progress in Mathematics **140**, Birkhäuser, Boston, 1996.